

Hadamard States for the Vector Potential on Asymptotically Flat Spacetimes

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Abstract. We develop a quantization scheme for the vector potential on globally hyperbolic spacetimes which realizes it as a locally covariant conformal quantum field theory. This result allows us to employ on a large class of backgrounds, which are asymptotically flat at null infinity, a bulk-to-boundary correspondence procedure in order to identify for the underlying field algebra a distinguished ground state which is of Hadamard form.

1 Introduction

A key feature of all free quantum field theories on Minkowski spacetime is covariance under the action of the underlying isometries, namely the Poincaré group. At a quantum level, Poincaré invariance further leads to the identification of a unique quasifree, pure, ground state [25]. Yet, as soon as we consider a non-trivial but fixed background, all these peculiarities disappear – no matter which free quantum field theory we consider – since the underlying isometry group is, in general, much smaller than the Poincaré group. As a result of this, it is not only impossible to select a preferred ground state on a curved background but one also has to be more careful when deciding whether a certain state is physically sensible or not. Nevertheless, in the past few years it has been become universally accepted that an answer to this query lies in the requirement that a physical state should be of Hadamard form, *i.e.*, it satisfies a specific condition on the singular structure of its two-point function [34, 35]. The main disadvantage of this paradigm is that, while it is extremely appealing from a mere mathematical point of view, it is rather hard to come up with a scheme to construct any such state, unless the underlying background is rather special, such as for example a globally hyperbolic and static spacetime [36].

A potential way to circumvent this problem has been put forth in the past five years and it calls for employing a so-called bulk-to-boundary reconstruction technique. According to this method, if a spacetime possesses a distinguished codimension 1 null submanifold, it is possible to associate to the latter an intrinsic $*$ -algebra on which one can try to encode the information of the bulk theory. More precisely, one has to construct an injective $*$ -homomorphism between the algebra of observables of the chosen bulk free field theory and the boundary theory. This has a twofold advantage: Every state for the boundary algebra induces automatically via the injection map a bulk counterpart whose properties can be studied and it is much easier to explicitly construct states on a

three-dimensional null manifold than to construct them directly in the bulk spacetime. As a matter of fact, this idea has already been successfully applied to construct distinguished Hadamard states for the massless scalar field on asymptotically flat spacetimes [13], and for both scalar and Dirac fields on a large class of cosmological backgrounds [9, 14, 15].

In this paper we will focus instead on the electromagnetic field. Although its dynamics, encoded in Maxwell's equations, is best described in terms of a two-form, the field strength or Faraday tensor, we will focus our attention to those scenarios where such a tensor can be written as the exterior derivative of a one-form, the vector potential. Our chief goal will be to show that it is possible to explicitly construct Hadamard states employing the bulk-to-boundary procedure just outlined if the underlying background is asymptotically flat and globally hyperbolic. It is worth mentioning that, up to now, there are no explicit examples of Hadamard states for the vector potential on a non-trivial background and even their existence has only recently been proven in [17] although under some additional restrictive hypotheses on the topology of the underlying background. The reason for such deficiency cannot only be traced to the intrinsic difficulty of constructing Hadamard states, but also to the peculiarity of the vector potential. Indeed, even in Minkowski spacetime and contrary to what happens with the scalar or with the Dirac field, there are problems arising with the standard quantization procedure of the vector potential [40, 41] which are at least partially overcome by employing the so-called Gupta-Bleuler formalism [4, 24]. We will not enter into a detailed analysis of this method but suffice it to say that recasting it on a curved background is a rather daunting task. In this paper we will show that using a complete gauge fixing together with the bulk-to-boundary procedure provides an alternative method which is applicable also on a wider class of curved backgrounds.

The paper will be organized as follows: In section 2, we will introduce our main notations and conventions and we will recollect the definition of an asymptotically flat spacetime with future timelike infinity. In particular, we will summarize the main geometric properties of the conformal boundary emphasizing mostly the role of the BMS group and the properties of intrinsicness and universality of null infinity in section 2.4. Section 3 focuses on the vector potential on curved backgrounds and section 3.1 is fully devoted to the analysis of its classical dynamics following mostly the earlier works of [8, 16, 20, 31]. We present our first novel result in section 3.2. Namely, we will show that the vector potential can be described as a locally covariant conformal quantum field theory thus extending the result of [8]. Furthermore, we will define the field algebra of this physical system. Section 4 is fully focused on the construction of quasi-free Hadamard states for the vector potential on a large class of asymptotically flat spacetimes. In particular, we will introduce the bulk-to-boundary projection technique in section 4.1. We will prove that it can be applied to the vector potential although a residual gauge freedom has to be accounted for in order to match the bulk algebra to a subalgebra of the boundary counterpart. Hence, in section 4.2 we construct explicitly a distinguished state for the boundary algebra proving that it induces a bulk state which is both Hadamard and invariant under bulk isometries. Furthermore, we show that the boundary state can be slightly modified in order to define a one-parameter family of states on null infinity which fulfil an exact KMS condition. The bulk counterpart turns out to enjoy the Hadamard property too. Eventually in section 5 we draw our conclusions.

2 Preliminaries

2.1 Notations and Conventions

On account of the many technical tools we shall use throughout the paper we deem more appropriate to introduce the main ones in the following paragraphs so that the reader can consult all our conventions at his leisure.

We shall call *spacetime* (M, g) , a four dimensional, Hausdorff, arcwise connected, smooth manifold M endowed with a smooth Lorentzian metric of signature $(-, +, +, +)$. As proven by Geroch [21, 22], this suffices to guarantee that M is paracompact and second countable. On top of the geometric structure we shall consider $\Omega^p(M, \mathbb{K})$ and $\Omega_0^p(M, \mathbb{K})$, respectively the set of smooth and of smooth and compactly supported p -forms on M with values in the field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . In the case that $\mathbb{K} = \mathbb{R}$ we often omit its explicit mention since no confusion can arise.

On these spaces one can define two natural operators: the *external derivative* d and the *Hodge dual* $*$. Notice that, whereas d is completely independent from g , $*$ is the unique isomorphism $\Omega_0^p(M, \mathbb{K}) \rightarrow \Omega_0^{4-p}(M, \mathbb{K})$ built upon the metric such that $\omega \wedge * \eta = g^{-1}(\omega, \eta) \mu_g$ for all $\omega, \eta \in \Omega^p(M, \mathbb{K})$, where $g^{-1}(\omega, \eta)$ and μ_g are natural pairing between p -forms and the volume form induced by the metric tensor g respectively. Furthermore, we can introduce a third operator, known as the *codifferential* $\delta \doteq * d *$: $\Omega^p(M, \mathbb{K}) \rightarrow \Omega^{p-1}(M, \mathbb{K})$, which is the formal adjoint of d with respect to the metric L^2 -pairing $\int(\cdot \wedge * \cdot)$. We can then combine d and δ to define the (formally self-adjoint) *Laplace-de Rham operator* $\square \doteq d \circ \delta + \delta \circ d$.

In the main body of the paper we will be often interested in (compactly supported) smooth forms which are either closed or coclosed. Therefore, we introduce the following novel notation:

$$\begin{aligned}\Omega_{(0),\delta}^p(M, \mathbb{K}) &\doteq \{\omega \in \Omega_{(0)}^p(M, \mathbb{K}) \mid \delta\omega = 0\}, \\ \Omega_{(0),d}^p(M, \mathbb{K}) &\doteq \{\omega \in \Omega_{(0)}^p(M, \mathbb{K}) \mid d\omega = 0\}.\end{aligned}$$

Another ingredient which we shall need in the forthcoming discussion is $H^p(M)$, the p -th *de Rham cohomology group* of M , – see *e.g.* [5] for a definition and a recollection of the main properties. It is noteworthy that, since these groups are built only out of the external derivative, they are homotopy invariants and, in particular, completely independent from a metric structure.

In the next section, we want to describe the dynamical behaviour of an electromagnetic field without sources on a possibly curved spacetime (M, g) . Therefore, we have to make sure that it is possible to define an initial value problem for Maxwell's equations and, to this avail, we require (M, g) to be *globally hyperbolic*. To wit, there exists Σ , a closed achronal subset of M whose domain of dependence coincides with the whole manifold. As proven in [2, 3], this suffices to guarantee that Σ can be chosen as a three dimensional smooth embedded hypersurface, called *Cauchy surface*, and that M is, moreover, isometric to the smooth product manifold $\mathbb{R} \times \Sigma$.

Since the Laplace-de Rham operator \square is normally hyperbolic, we can find unique retarded and advanced fundamental solutions $G^\pm : \Omega_0^p(M, \mathbb{K}) \rightarrow \Omega^p(M, \mathbb{K})$ such that $\square \circ G^\pm = G^\pm \circ \square = \text{id}$ and $\text{supp } G^\pm(f) \subseteq J^\mp(\text{supp } f)$ as shown in [1, Chap. 3]. Further, we define the *causal propagator* $G \doteq G^+ - G^-$ as the difference between the retarded and advanced fundamental solution.

2.2 Conformal Transformations

Let us consider two arbitrary spacetimes (M, g) and $(\widetilde{M}, \tilde{g})$. Following partly [42, Appendix C & D], we say that a map $\psi : M \rightarrow \widetilde{M}$ is a *conformal embedding* if it is smooth, injective and if there exists a strictly positive function $\Xi \in C^\infty(M)$ such that $\psi^*\tilde{g} = \Xi^2 g$. If ψ is even a diffeomorphism, then it is referred to as *conformal isometry*.

In order to understand the effect of such ψ on classical fields, we recall that these are best understood as sections of a suitable vector bundle. Let us thus choose a vector bundle $\pi : E \rightarrow M$. A conformal isometry $\psi : M \rightarrow \widetilde{M}$ yields automatically on M the pull-back bundle ψ^*E , so that, if we refer to $\Gamma(E)$ as the space of smooth sections, an element $\tilde{s} \in \Gamma(E)$ induces $s = \psi^*\tilde{s} \in \Gamma(\psi^*E)$. Furthermore, we say that a section $s' \in \Gamma(\psi^*E)$ transforms with *conformal weight* w if $s' = \Xi^{-w}\psi^*\tilde{s}$.

In the forthcoming discussion of the vector potential, an important role will be played by the behaviour of the operators d , δ and \square under conformal isometries:

Proposition 2.1. *Let (M, g) , $(\widetilde{M}, \tilde{g})$ be arbitrary spacetimes so that $\psi : M \rightarrow \widetilde{M}$ is a conformal isometry with $\psi^*\tilde{g} = \Xi^2 g$. Further, let the codifferential on M and \widetilde{M} be called δ and $\tilde{\delta}$ respectively. Then, it holds that*

$$\psi^*\tilde{\delta}\tilde{\phi} = \Xi^{-2}(\delta\phi - 2g^{-1}(\Upsilon, \phi)), \quad (2.1a)$$

$$\psi^*\tilde{\delta}\tilde{\phi} = \Xi^{-4}\delta\varphi, \quad (2.1b)$$

$$\psi^*\tilde{\delta}d\tilde{\phi} = \Xi^{-2}\delta d\phi, \quad (2.1c)$$

$$\psi^*\tilde{\square}\tilde{\phi} = \Xi^{-2}(\square\phi - 2dg^{-1}(\Upsilon, \phi) + 2\Upsilon \wedge (2g^{-1}(\Upsilon, \phi) - \delta\phi)), \quad (2.1d)$$

where $\phi \doteq \Xi^{-2}\varphi \doteq \psi^*\tilde{\phi}$, $\Upsilon \doteq \Xi^{-1}d\Xi$ and $g^{-1}(\cdot, \cdot)$ is the metric pairing of 1-forms.

Proof. Since the exterior derivative depends only on the differentiable structure of the manifold, it holds that $\psi^*d\omega = d\psi^*\omega$ for any $\omega \in \Omega^p(\widetilde{M})$. On the contrary, the Hodge operator is built out of the metric and thus, on account of its definition,

$$\psi^*\tilde{*}\omega = \Omega^{4-2p} * \psi^*\omega \quad (2.2)$$

for all $\omega \in \Omega^p(\widetilde{M})$. If we put these ingredients together with the properties of the pull-back of forms and with $\delta = *d*$, (2.1a) descends immediately.

Let us now start from (2.1a) and replace ϕ with φ . Then, the right hand side of (2.1a) reads $\Xi^{-2}(\delta\Xi^{-2}\varphi - 2g^{-1}(\Upsilon, \Xi^{-2}\varphi))$. An application of the identity $\delta(f\omega) = f\delta\omega - g^{-1}(df, \omega)$ for all $f \in C^\infty(M)$ and $\omega \in \Omega^1(M)$ yields (2.1b).

To prove the third equation, we stress that the properties of d and $*$ under a conformal isometry entail that $\psi^*(d\tilde{*}d\tilde{\phi}) = d*d\phi$. Since $d\tilde{*}d\tilde{\phi} \in \Omega^3(M)$, (2.1c) descends immediately from (2.2) with $p = 3$.

In order to show equality in (2.1d), recall that $\tilde{\square} = \tilde{\delta}d + d\tilde{\delta}$. Hence, we are left to evaluate $\psi^*(d\tilde{\delta}\tilde{\phi})$. On account of the properties of the exterior derivative and of the codifferential, the following chain of identities holds true:

$$\begin{aligned} \psi^*(d\tilde{\delta}\tilde{\phi}) &= d\psi^*(\tilde{\delta}\tilde{\phi}) = d(\Xi^{-2}(\delta\phi - 2g^{-1}(\Upsilon, \phi))) \\ &= \Xi^{-2}d\delta\phi - 2\Xi^{-2}\Upsilon \wedge \delta\phi - 2dg^{-1}(\Upsilon, \phi), \end{aligned}$$

where, in the second equality, we employed (2.1a). This result combined with (2.1c) yields the sought (2.1d). \square

On account of these results, we say that the codifferential δ and the operator δd are *conformally invariant* when acting on 1-forms transforming with conformal weight 0 and -2 respectively.

2.3 Asymptotically Flat Spacetimes

In this paper we will be mostly interested in a subclass of globally hyperbolic spacetimes (M, g) which are solutions of Einstein vacuum equations and whose behaviour at infinity along null geodesics mimics that of Minkowski spacetime. To wit, we consider an **asymptotically flat spacetime with future time infinity** i^+ , *i.e.*, a time-oriented globally hyperbolic spacetime (M, g) , called *physical spacetime*, such that there exists a second globally hyperbolic spacetime $(\widetilde{M}, \tilde{g})$, called *unphysical spacetime*, with a preferred point i^+ , a diffeomorphism $\psi : M \rightarrow \psi(M) \subset \widetilde{M}$ and a function $\Xi : \psi(M) \rightarrow (0, \infty)$ so that $\tilde{g} = \Xi^2 \psi_* g$. Moreover, the following requirements are satisfied:

- a) If we call $J^-(i^+)$ the causal past of i^+ , this is a closed set such that $\psi(M) = J^-(i^+) \setminus \partial J^-(i^+)$ and we have $\partial M = \partial J^-(i^+) = \mathcal{I}^+ \cup \{i^+\}$, where \mathcal{I}^+ is *future null infinity*.
- b) Ξ can be extended to a smooth function on the whole \widetilde{M} and it vanishes on $\mathcal{I}^+ \cup \{i^+\}$. Furthermore, $d\Xi \neq 0$ on \mathcal{I}^+ while $d\Xi = 0$ on i^+ and $\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Xi = -2\tilde{g}_{\mu\nu}$ at i^+ .
- c) If $n^\mu \doteq \tilde{\nabla}^\mu \Xi$, there exists a smooth and positive function ξ supported at least in a neighbourhood of \mathcal{I}^+ such that $\tilde{\nabla}_\mu (\xi^4 n^\mu) = 0$ on \mathcal{I}^+ and the integral curves of $\xi^{-1} n$ are complete on future null infinity.

Here the connection $\tilde{\nabla}$ is the Levi-Civita connection built out of \tilde{g} .

It is worth remarking that this definition is different from the standard one (cf. *e.g.* [42, Chap. 11]), where asymptotic flatness is defined with respect to i_0 , spatial infinity, as distinguished point. The underlying reason for adopting such a choice is related to our need to describe solutions of a second order hyperbolic partial differential operator on such backgrounds. If smooth and compactly supported initial data are assigned, the associated solutions will be supported in the causal future and past of the initial data and thus it is important that either i^+ or i^- are included in order to avoid any loss of information. The above definition appeared already in [29, 30], where the equivalence with the definition proposed by Friedrich (see [18] and references therein) is also pointed out. Notice that, in comparison with these last cited papers, we dropped the request of strong causality for M and \widetilde{M} since it is a property enjoyed by all globally hyperbolic spacetimes. Furthermore, although one could equivalently define an asymptotically flat spacetime with past time infinity i^- , we will not mention this possibility again since all our results can be translated slavishly to that case.

2.4 Geometric Properties of Null Infinity

For our purposes the most notable property of an asymptotically flat spacetime is the existence of future null infinity which is a three dimensional submanifold of \widetilde{M} generated by the null geodesics emanating from i^+ , *i.e.*, the integral curve of n . For this reason \mathcal{I}^+ is diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$ although the possible metric structures are affected

by the existence of a gauge freedom which corresponds to the rescaling of Ξ to $\xi\Xi$, where ξ is a smooth function which is strictly positive in $\psi(M)$ and a neighbourhood of \mathcal{I}^+ . Furthermore, if we introduce for any fixed asymptotically flat spacetime (M, g) the set C composed of equivalence classes of triples (\mathcal{I}^+, h, n) , where $h \doteq \tilde{g} \upharpoonright_{\mathcal{I}^+}$ and $(\mathcal{I}^+, h, n) \sim (\mathcal{I}^+, \xi^2 h, \xi^{-1} n)$ for any choice of ξ satisfying c), there is no physical mean to select a preferred element in C . This is called the *intrinsicness* of \mathcal{I}^+ which enlightens the relevance of the boundary structure together with the property of *universality*. To wit, if we select any pair of asymptotically flat spacetimes, (M_1, g_1) and (M_2, g_2) , together with the corresponding triples, say $(\mathcal{I}_1^+, h_1, n_1)$ and $(\mathcal{I}_2^+, h_2, n_2)$, there always exists a diffeomorphism $\gamma : \mathcal{I}_1^+ \rightarrow \mathcal{I}_2^+$ such that $h_1 = \gamma^* h_2$ and $n_2 = \gamma_* n_1$. Although we leave a reader interested in the proof of this last statement to [42, Chap. 11], it is noteworthy that it relies on a very important additional property of an asymptotically flat spacetime: In each equivalence class there exists a choice of conformal gauge ξ_B yielding a coordinate system $(u, \Xi, \theta, \varphi)$ in a neighbourhood of \mathcal{I}^+ , called a *Bondi frame*, and the (rescaled) unphysical metric tensor becomes

$$\tilde{g} \upharpoonright_{\mathcal{I}^+} = -2 du d\Xi + d\theta^2 + \sin^2 \theta d\varphi^2.$$

That is, Ξ is promoted from conformal factor to a coordinate, thus indicating that future null infinity is the locus $\Xi = 0$, while u is the affine parameter of the null geodesics generating \mathcal{I}^+ . Thus, at each point on \mathcal{I}^+ the vector field n coincides with $\partial/\partial u$.

A very important role is played by the subgroup of diffeomorphisms of \mathcal{I}^+ which maps any triple (\mathcal{I}^+, h, n) into a gauge equivalent one. This is the so-called **Bondi-Metzner-Sachs** (BMS) group of transformations whose structure is that of semidirect product, *i.e.*, $BMS = SO_0(3, 1) \ltimes C^\infty(\mathbb{S}^2)$. The action of each element $\gamma \in SO_0(3, 1) \ltimes C^\infty(\mathbb{S}^2)$ is best represented in a Bondi frame, that is, going over to the complex stereographic coordinate $z = e^{i\varphi} \cot \theta/2$, we have the mapping

$$\begin{aligned} u &\mapsto u' = K_\gamma(z, \bar{z})(u + \alpha_\gamma(z, \bar{z})), \\ z &\mapsto z' = \frac{az + b}{cz + d} \quad \text{and c.c.}, \end{aligned} \tag{2.3}$$

where

$$K_\gamma(z, \bar{z}) = \frac{1 + |z|^2}{|az + b|^2 + |cz + d|^2}, \quad \alpha_\gamma \in C^\infty(\mathbb{S}^2),$$

and $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$, *i.e.*, the $SO_0(3, 1) \cong PSL(2, \mathbb{C}) \cong \text{Aut}(\mathbb{C} \cup \{\infty\})$ part is represented via Möbius transformations.

On account of the the representation of the BMS group in stereographic coordinates, we can now consider a family of representations of the BMS group on sections of a vector bundle $\pi : E \rightarrow \mathcal{I}^+$. Namely, for each $w \in \mathbb{R}$ we have the representation $\Pi^w : \Gamma(E) \rightarrow \Gamma(E)$ defined by

$$(\Pi_\gamma^w s)(u', z', \bar{z}') \doteq K_\gamma(z, \bar{z})^w s(u + \alpha_\gamma(z, \bar{z}), z, \bar{z}), \tag{2.4}$$

where K_γ, α_γ as above, for each $\gamma \in BMS$ and all $s \in \Gamma(E)$.

From the geometrical point of view the BMS group can be interpreted as the group of asymptotic symmetries for all asymptotically flat spacetimes as shown in [42, Chap. 11]. This entails that each one-parameter subgroup of diffeomorphisms of \mathcal{I}^+ generated by a smooth vector field X' is a subgroup of the BMS group if and only if X' can be

extended smoothly but not necessarily in a unique way to a vector field X of the bulk spacetime (M, g) in such a way that $\Xi^2 \mathcal{L}_X g$ admits a smooth and vanishing extension to \mathcal{I}^+ . In particular, as proven by Geroch in [23], if X is a Killing vector field in the physical spacetime, this guarantees the existence of a unique extension to future null infinity. Thus, it is commonly said that the BMS group encodes the information of all possible bulk symmetries.

3 The Vector Potential on Curved Spacetimes

In this section we will introduce the main object of our studies, the vector potential. We will discuss its classical kinematical and dynamical configurations and outline its quantization in the algebraic approach. We will emphasize in particular how this physical system can be described in terms of a locally covariant conformal quantum field theory. Part of the material we shall present has already appeared in the literature and, in particular, we will benefit of earlier analyses [8, 11, 16, 17, 31].

3.1 Classical Dynamics

Classical electromagnetism on a globally hyperbolic spacetime (M, g) is assumed to be described by the natural extension to curved backgrounds of the manifestly covariant version of Maxwell's equations on Minkowski spacetime. In other words, one should consider the so-called field strength tensor $F \in \Omega^2(M)$, which satisfies

$$dF = 0, \quad \delta F = -j,$$

where the source j is a coclosed smooth 1-form. In the forthcoming discussion we shall always assume that $j = 0$ and hence we only consider the Maxwell equations without sources.

The dynamics provided by this system is best analysed in terms of the vector potential, that is, $A \in \Omega^1(M)$ such that $F = dA$. If $H^2(M)$, the second de Rham cohomology group of M , is trivial, the global existence of this 1-form is a by-product of the Poincaré lemma. Otherwise there exist field strength tensors which solve Maxwell's equations and do not descend from a vector potential. Since the goal of this paper is to focus on the construction of Hadamard states for A , we shall not dwell on this problem. Nevertheless, the reader is warned of this obstruction and, if interested in a discussion of this issue, should refer to [11]. Hence, in terms of the vector potential, the dynamics is ruled by

$$\delta dA = 0. \tag{3.1}$$

In an arbitrary coordinate system of (M, g) this reads

$$-\nabla_\nu \nabla^\nu A_\mu + \nabla_\mu \nabla^\nu A_\nu + R_\mu^\nu A_\nu = 0, \tag{3.2}$$

where $R_{\mu\nu}$ is the Ricci tensor built out of g . Inspecting the principal symbol of (3.2), one can realize that the differential operator is not of hyperbolic type and thus one cannot straightforwardly set up an initial value problem. This quandary can be solved if we recall that two vector potentials can yield the same field strength, *i.e.*, they are *gauge equivalent*, and thus they should be identified. The ensuing equivalence classes are

$$[A] = \{A' \in \Omega^1(M) \mid A' \sim A, \text{ iff } \exists \Lambda \in \Omega_d^1(M) \text{ with } A - A' = \Lambda\}.$$

That is, two vector potentials are gauge equivalent and, hence, in the same equivalence class if they differ by a closed 1-form. As proven for example in [8], the following lemma holds true:

Lemma 3.1. *Each 1-form A' satisfying $\delta dA' = 0$ is gauge equivalent to a vector potential A such that*

$$\square A = 0, \quad \delta A = 0. \quad (3.3)$$

In an arbitrary coordinate system (3.3) reads $-\nabla_\nu \nabla^\nu A_\mu + R_\mu^\nu A_\nu = 0$ together with $\nabla^\mu A_\mu = 0$. In other words, the dynamics of A is ruled by a second order hyperbolic partial differential operator with principal symbol of metric type and the vector potential is further constrained to satisfy the *Lorenz gauge*. Therefore, each solution with smooth and compactly supported initial data for (3.3), dubbed *Lorenz solution*, can be written as $A = Gf$ where $f \in \Omega_{0,\delta}^1(M)$ and G is the causal propagator associated to the Laplace-de Rham operator \square . Notice that the coclosedness of f suffices to guarantee that A fulfils the Lorenz gauge because, as already remarked in [16, 17, 31], $G \circ \delta = \delta \circ G$, where, on the left hand side, G is meant as the causal propagator for \square acting on p -forms whereas, on the right hand side, it is the one acting on $(p-1)$ -forms.

We also have to take into account that there could exist two Lorenz solutions which are gauge equivalent and hence yield the same field strength tensor. This comes to no surprise since, as often remarked in the quantization of the electromagnetic field on Minkowski spacetime, the Lorenz gauge is incomplete, namely it still leaves us with a residual gauge freedom. The following proposition clarifies under which conditions on the initial data this can happen.

Proposition 3.2. *Let $f, f' \in \Omega_{0,\delta}^1(M)$. Then $Gf \sim Gf'$ if and only if there exists $\lambda \in \Omega_{0,d}^2(M)$ such that $f - f' = \delta\lambda$.*

Proof. Suppose $f - f' = \delta\lambda$ for $f, f' \in \Omega_{0,\delta}^1(M)$. Then $\Lambda \doteq Gf - Gf' = G(\delta\lambda)$ is such that

$$d\Lambda = dG(\delta\lambda) = G(d\delta\lambda) = G(\square\lambda) = 0,$$

where we employed the definition of the Laplace-de Rham operator and closedness of λ in the second equality.

Conversely, suppose that two initial data $f, f' \in \Omega_{0,\delta}^1(M)$ are such that the associated Lorenz solutions are gauge equivalent. Therefore, $G(f - f')$ is closed, *i.e.*, $G(d(f - f')) = 0$, which in turn guarantees the existence of $\lambda \in \Omega_0^2(M)$ such that $d(f - f') = \square\lambda$. If one applies to both sides the external derivative, the ensuing identity, $d\square\lambda = \square d\lambda = 0$, entails $d\lambda = 0$ as λ is compactly supported. Furthermore,

$$\square\delta\lambda = \delta\square\lambda = \delta d(f - f') = \square(f - f'),$$

where, in the last equality, we exploited the coclosedness of both f and f' . Since all forms involved are compactly supported, the above chain of identities yields that $\delta\lambda = f - f'$. \square

We denote the collection of these equivalence classes of test forms associated to Lorenz solutions

$$\mathcal{S}(M) \doteq \frac{\Omega_{0,\delta}^1(M)}{\delta\Omega_{0,d}^2(M)}.$$

Notice then, that the bottom line of proposition 3.2 is that $\mathcal{S}(M)$ is in one-to-one correspondence with the gauge equivalence classes of solutions of (3.1).

Although proposition 3.2 applies to all globally hyperbolic spacetimes, it is possible to derive a more stringent result if some additional assumption on the topology of (M, g) are made. As we shall see, this will play an important role in the discussion of the quantization of the theory.

Corollary 3.3. *Let (M, g) be a globally hyperbolic spacetime such that either $H^1(M)$ or $H^2(M)$ is trivial. Then, two Lorenz solutions Gf and Gf' are equivalent if and only if there exist $\alpha \in \Omega_0^1(M)$ and $\chi \in C^\infty(M)$ such that*

$$f - f' = \delta d\alpha, \quad G(f - f') = d\chi.$$

Proof. Let us start supposing $H^1(M) = \{0\}$. Then two gauge equivalent Lorenz solutions Gf and Gf' differ by a closed one form, which thus suffices to guarantee the existence of $\chi \in C_0^\infty(M)$ such that $G(f - f') = d\chi$. Furthermore, since $\delta G(f - f') = 0$, it holds that $\square\chi = 0$ which implies that there exists $\eta \in C_0^\infty(M)$ such that $G\eta = \chi$. Hence, $G(f - f' - d\eta) = 0$ and one can find $\alpha \in \Omega_0^1(M)$ such that $f - f' = d\eta - \square\alpha$. Since both f and f' are coclosed, also $\square(\eta - \delta\alpha)$ vanishes, which is tantamount to $\eta = \delta\alpha$ and, thus, to $f - f' = d\delta\alpha$ which is the sought identity.

Conversely, suppose now that $H^2(M)$ is trivial. On account of proposition 3.2 we know that two Lorenz solutions Gf and Gf' are gauge equivalent if $f - f' = \delta\lambda$ with $\lambda \in \Omega_{0,d}^2(M)$. Hence, there exists $\alpha \in \Omega_0^1(M)$ such that $\lambda = d\alpha$ and we obtain $f - f' = \delta d\alpha$. If we apply to both sides the causal propagator, we end up with $G(f - f') = G(\delta d\alpha) = -G(d\delta\alpha) = -dG(\delta\alpha)$ which is the sought result if we set $\chi = -G(\delta\alpha)$. \square

According to corollary 3.3, we obtain

$$\mathcal{S}(M) \cong \frac{\Omega_{0,\delta}^1(M)}{\delta d\Omega_0^1(M)}$$

whenever either $H^1(M)$ or $H^2(M)$ is trivial. To conclude our analysis of the classical theory of the vector potential, we need a last result which gives $\mathcal{S}(M)$ the structure of a phase space and, at the same time, creates a bridge towards the quantization of the theory.

Proposition 3.4. *The set $\mathcal{S}(M)$ is a weakly non-degenerate symplectic space if either $H^1(M)$ or $H^2(M)$ is trivial and if endowed with the antisymmetric bilinear form $\sigma : \mathcal{S}(M) \times \mathcal{S}(M) \rightarrow \mathbb{R}$,*

$$\sigma([f], [h]) \doteq G(f \otimes h) = \int_M Gf \wedge *h. \quad (3.4)$$

Here, $[f], [h] \in \mathcal{S}(M)$ with $f, h \in \Omega_{0,\delta}^1(M)$ being two arbitrary representatives of these equivalence classes.

Proof. As a first step we prove that (3.4) is well-defined, i.e., it is independent from the chosen representative. Let $h, h' \in [h]$. Then the following chain of identities holds true:

$$\sigma([f], [0]) = \int_M Gf \wedge *(h - h') = \int_M Gf \wedge *\delta d\alpha = \int_M \delta dGf \wedge *\alpha = 0,$$

where in the second equality we exploited corollary 3.3 while in the last we used the properties of G and the coclosedness of f . Since σ is bilinear and antisymmetric this suffices to prove the independence of both entries from the chosen representative.

To prove weak non-degenerateness, it suffices to observe that $\ker G \cap \Omega_{0,\delta}^1(M) = \delta d\Omega_0^1(M)$ and that the metric pairing of 1-forms in the integrand is non-degenerate. \square

It is worthwhile to remark that this last proposition extends the results of [8], where the same statement was proven under the hypothesis that $H^1(M)$ is trivial. If the underlying manifold does not fulfil the assumptions on the topology, it turns out that it is not clear whether (3.4) is independent from the choice of the representative of the equivalence classes involved. Without such a result, as it will be clear from the procedure outlined in the next subsection, it is prohibitive to associate to the vector potential a genuine field algebra. Notice that it is easy to construct globally hyperbolic spacetimes which do not fulfil the hypotheses of proposition 3.4. For example, any ultrastatic manifold (M, g) isometric to $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^2$ with line element $ds^2 = -dt^2 + h$, where t is a global time-coordinate and h is a smooth, Riemannian and time-independent metric on the Cauchy surface $\mathbb{S}^1 \times \mathbb{S}^2$, has non-trivial first and second de Rham cohomology group.

Let us conclude this subsection by investigating some of the properties of the vector potential under conformal transformations. As a first result we have the following corollary which is a direct consequence of (2.1c).

Corollary 3.5. *Maxwell's equation (3.1) is conformally invariant on 1-forms of conformal weight 0. That is, given two conformally isometric spacetimes (M, g) , (\tilde{M}, \tilde{g}) such that $\psi : M \rightarrow \tilde{M}$ with $\psi^*\tilde{g} = \Xi^2 g$, then $A \in \Omega^1(M)$ is a solution of (3.1) in (M, g) if and only if $\tilde{A} \in \Omega^1(\tilde{M})$ solves (3.1) in (\tilde{M}, \tilde{g}) and $A = \psi^*\tilde{A}$.*

Notice that the properties of Lorenz solutions under conformal transformations on a special class of curved background were also emphasized in [33].

In this subsection we studied the equation (3.1) by considering the equivalence classes of Lorenz solutions, *i.e.*, solutions of the constrained hyperbolic system (3.3). The Lorenz gauge condition, however, is not conformally invariant on 1-forms of conformal weight 0 as it can be readily seen from (2.1a). Nevertheless, we can obtain the following important result.

Proposition 3.6. *Let (M, g) , (\tilde{M}, \tilde{g}) be globally hyperbolic spacetimes such that $\psi : M \rightarrow \tilde{M}$ is a conformal isometry with $\psi^*\tilde{g} = \Xi^2 g$. Further, let $\tilde{A} = \tilde{G}\tilde{f}$ with $\tilde{f} \in \Omega_{0,\delta}^1(\tilde{M})$ be a solution of (3.3) in (\tilde{M}, \tilde{g}) (*i.e.* \tilde{G} is the causal propagator associated to the Laplace-de Rham operator $\tilde{\square}$ in (\tilde{M}, \tilde{g})). Then,*

$$A = Gf = G\Xi^2\psi^*\tilde{f} \quad \text{with} \quad f = \Xi^2\psi^*\tilde{f} \in \Omega_{0,\delta}^1(M)$$

solves (3.3) in (M, g) and $\psi^\tilde{A} = A + d\lambda$ for some $\lambda \in C^\infty(M)$.*

Proof. We obtain via (2.1c)

$$\psi^*\tilde{f} = \psi^*\tilde{\delta}\tilde{A}^\pm = \Xi^{-2}\delta d\psi^*\tilde{A}^\pm = \Xi^{-2}f,$$

where $\tilde{A}^\pm = \tilde{G}^\pm\tilde{f}$ and $f \in \Omega_{0,\delta}^1(M)$. Moreover, according to (2.1a), we have $0 = \psi^*\tilde{\delta}\tilde{A}^\pm = \Xi^{-2}(\delta\psi^*\tilde{A}^\pm - 2g^{-1}(\Upsilon, \psi^*\tilde{A}^\pm))$. Taking the exterior derivative, this yields

$d\delta\psi^*\tilde{A}^\pm = 2dg^{-1}(\Upsilon, \psi^*\tilde{A}^\pm)$ and thus $\square\psi^*\tilde{A}^\pm = f + 2dg^{-1}(\Upsilon, \psi^*\tilde{A}^\pm)$. Applying the advanced resp. the retarded Green's operator and subtracting both results, we finally obtain

$$\psi^*\tilde{A} = A + 2d(G^-g^{-1}(\Upsilon, \psi^*\tilde{A}^-) - G^+g^{-1}(\Upsilon, \psi^*\tilde{A}^+))$$

and therefore $\psi^*\tilde{A} = A + d\lambda$. \square

That is, if the vector potential is to transform with conformal weight 0, the corresponding test form has to transform with conformal weight -2 .

Corollary 3.7. *Let (M, g) , $(\widetilde{M}, \tilde{g})$ and ψ . For any $\tilde{f}, \tilde{h} \in \Omega_{0,\delta}^1(\widetilde{M})$ we have*

$$G(f \otimes h) = \tilde{G}(\tilde{f} \otimes \tilde{h}),$$

where $f = \Xi^{-2}\psi^*\tilde{f}$, $h = \Xi^{-2}\psi^*\tilde{h}$ and G, \tilde{G} are the causal propagator on (M, g) , $(\widetilde{M}, \tilde{g})$ respectively. That is, the symplectic form defined in proposition 3.4 is conformally invariant.

Proof. Denoting by $\tilde{*}$ the Hodge operator associated to \tilde{g} , we obtain

$$\tilde{G}(\tilde{f} \otimes \tilde{h}) = \int_M \psi^*(\tilde{G}\tilde{f} \wedge \tilde{*}\tilde{h}) = \int_M \psi^*(\tilde{G}\tilde{f}) \wedge *h = \int_M (Gf + d\lambda) \wedge *h$$

since $\psi^*(\tilde{G}\tilde{f}) = Gf + d\lambda$ for some $\lambda \in C^\infty(M)$ by proposition 3.6. Furthermore, the second term, $\int_M d\lambda \wedge *h$, vanishes because h is coclosed. \square

3.2 Quantization of the Vector Potential

The study of the classical dynamics of (3.1) via (3.3) allows us as a by-product to tackle the problem of quantizing the underlying theory. We shall approach this issue in the algebraic formalism via a two-step approach. First, we associate to the field under investigation a suitable $*$ -algebra of observables which is compatible with the dynamics. Then, in the second step, we endow the resulting algebra with a suitable (algebraic) quantum state. We will leave this problem to the next section, while, in the present, we identify the so-called field algebra and we show that its structure allows us to describe the vector potential as a locally covariant conformal quantum field theory along the lines of [7] and [32].

In order to inquire whether the set of solutions of (3.1) or equivalently of (3.3) can be studied as a locally covariant conformal quantum field theory we need to introduce some additional mathematical tools. We introduce the following categories:

- **\mathcal{EMan}** : the category whose objects are oriented and time-oriented globally hyperbolic spacetimes (M, g) . The arrows are conformal embeddings $\psi : M \hookrightarrow M'$ which preserve orientation and time-orientation and are such that $\psi(M)$ is a causally convex, hence globally hyperbolic, subset of \widetilde{M}' . The composition of morphisms is that of smooth maps and the unit element is the identity map.
- **\mathcal{EMan}'** : the subcategory of **\mathcal{EMan}** whose objects are oriented and time-oriented globally hyperbolic spacetimes (M, g) with either $H^1(M) = \{0\}$ or $H^2(M) = \{0\}$. The arrows are the same as those of **\mathcal{EMan}** .

- $\ast\text{-}\mathcal{Alg}$: the category whose objects are topological unital \ast -algebras (over \mathbb{R} or \mathbb{C}) and whose morphisms are injective \ast -homomorphisms. Again the composition of morphisms is that of smooth maps and the unit element is the identity map.

Definition 3.8. A *locally covariant conformal quantum field theory* (LCCQFT) is a covariant functor $\mathcal{A} : \mathcal{CMan} \rightarrow \ast\text{-}\mathcal{Alg}$. The theory is called **causal** if for any two morphisms $\psi_1 : M_1 \rightarrow M$ and $\psi_2 : M_2 \rightarrow M$ of \mathcal{CMan} such that $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally separated

$$[\mathcal{A}(\psi_1)(\mathcal{A}(M_1, g_1)), \mathcal{A}(\psi_2)(\mathcal{A}(M_2, g_2))] = 0.$$

Furthermore, the theory is said to satisfy the **time-slice axiom** if $\mathcal{A}(\psi)(\mathcal{A}(M, g)) = \mathcal{A}(M', g')$ for all morphisms $\psi : M \rightarrow M'$ of \mathcal{CMan} such that $\psi(M)$ contains a Cauchy surface of M' .

In order to verify whether the vector potential fits in the scheme depicted by the last definition, we have to make sure that it is possible to associate to the vector potential a suitable \ast -algebra of observables. In the algebraic approach to quantum field theory on curved spacetimes, the standard procedure calls for identifying the so-called *field algebra* as a quotient of the Borchers-Uhlmann algebra with respect to the ideal generated by the equations of motion and the canonical commutation relations.

Hence, if we recall that the solutions of (3.1) have conformal weight 0 and that proposition 3.4 guarantees the existence of a weakly non-degenerate symplectic form for the space of classical solutions of (3.3) if (M, g) has either trivial first or second de Rham cohomology group we can write similarly to [32]:

Definition 3.9. We call **field algebra** of the vector potential on a globally hyperbolic spacetime (M, g) with $H^1(M) = \{0\}$ or $H^2(M) = \{0\}$ the \ast -algebra defined as the quotient $\mathcal{A}(M) \doteq \mathcal{A}(M)/\mathcal{I}(M)$. Here, $\mathcal{A}(M)$ is the free unital \ast -algebra generated by $\Omega_{0,\delta}^1(M, \mathbb{C})$, i.e.,

$$\mathcal{A}(M) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \Omega_{0,\delta}^1(M, \mathbb{C})^{\otimes n}.$$

At the same time $\mathcal{I}(M)$ is the closed \ast -ideal of $\mathcal{A}(M)$ generated by the elements

$$\delta d\Omega_0^1(M, \mathbb{C}) \quad \text{and} \quad -iG(f \otimes h) \oplus f \otimes h - h \otimes f,$$

for all $f, h \in \Omega_{0,\delta}^1(M, \mathbb{C})$, where G is the causal propagator to \square on (M, g) . The \ast -operation is complex conjugation.

Recalling proposition 3.6, we obtain the sought result, namely that field algebra yields a locally covariant conformal quantum field.

Proposition 3.10. The vector potential can be described as a locally covariant conformal quantum field theory $\mathcal{A} : \mathcal{CMan}' \rightarrow \ast\text{-}\mathcal{Alg}$ which assigns to each object (M, g) in \mathcal{CMan}' the \ast -algebra $\mathcal{A}(M)$ and to each conformal embedding $\psi : M \hookrightarrow M'$ with $\psi^\ast g' = \Xi^2 g$ the \ast -homomorphism $\alpha_\psi \doteq q_{M'} \circ \psi_\ast^{-2}$, where $\psi_\ast^{-2} \doteq \Xi^{-2} \psi_\ast$, while $q_{M'}$ is the projection from $\mathcal{A}(M')$ to $\mathcal{A}^0(M')$. Furthermore, \mathcal{A} is causal and satisfies the time-slice axiom.

Proof. According to definition 3.9 we can associate to (M, g) the conformal field algebra $\mathcal{A}(M)$. Picking any arrow ψ between (M, g) and another object $(M', g') \in \mathfrak{CMan}'$, we see that it is a conformal isometry from (M, g) into $(\psi(M), g|_{\psi(M)})$ which thus is also a proper map. In other words, $\Omega_0^1(M)$ is isomorphic to $\Omega_0^1(\psi(M))$. Since the codifferential is conformally invariant on 1-forms of conformal weight -2 , we also have an isomorphism $\Omega_{0,\delta}^1(M) \cong \Omega_{0,\delta}^1(\psi(M))$ via ψ_*^{-2} . This entails that $\psi_*^{-2}\mathcal{A}(M) = \mathcal{A}^0(\psi(M))$, *i.e.*, given for example $a = a_0 \oplus a_1 \oplus a_{2,1} \otimes a_{2,2} \oplus \dots \in \mathcal{A}(M)$, we obtain $\psi_*^{-2}a = a_0 \oplus \psi_*^{-2}a_1 \oplus \psi_*^{-2}a_{2,1} \otimes \psi_*^{-2}a_{2,2} \oplus \dots \in \mathcal{A}^0(\psi(M))$. Furthermore, since f transforms with conformal weight -2 by proposition 3.6 and since the symplectic form is conformally invariant by corollary 3.7, one sees that $\psi_*^{-2}\mathcal{J}(M) = \mathcal{J}(\psi(M))$. In order to understand that the map ψ_*^{-2} is also a $*$ -homomorphism, notice that all operations involve only real structures and thus all complex conjugations are left untouched by ψ_*^{-2} .

In order to extend the result to M' let us remember that $\psi(M) \subset M'$ and, thus, since all sections involved are compactly supported, we can extend each element of $\Omega_{0,\delta}^1(\psi(M))$ to 0 on all points of M' not lying in $\psi(M)$, *i.e.* $\Omega_{0,\delta}^1(\psi(M)) \subset \Omega_{0,\delta}^1(M')$. Consequently $\mathcal{A}^0(\psi(M)) \subset \mathcal{A}^0(M')$. This remark, together with the inclusion $\mathcal{J}(\psi(M)) \subset \mathcal{J}(M')$ and the definition of α_ψ via $q_{M'}$, suffices to conclude that α_ψ is a $*$ -homomorphism.

Causality is a direct by-product of the definition of the ideal $\mathcal{J}(M)$ via the causal propagator G . As a matter of fact, if we consider two arrows $\psi_i : M_i \rightarrow M$ with causally separated images and two elements $f \equiv 0 \oplus f \oplus 0 \oplus \dots \in \mathcal{A}(\psi(M_1)) \subset \mathcal{A}(M)$ and $h \equiv 0 \oplus h \oplus 0 \oplus \dots \in \mathcal{A}(\psi(M_2)) \subset \mathcal{A}(M)$, then $[f, h] = f \otimes h - h \otimes f = iG(f \otimes h) = 0$ on account of the support properties of the causal propagator. Since the chosen elements are generating $\mathcal{A}(\psi(M_1))$ and $\mathcal{A}(\psi(M_2))$ respectively, this suffices to draw the sought conclusion.

The time-slice axiom can be proven using a procedure similar to the one employed in [8]. Therefore let us pick any arrow $\psi : M \hookrightarrow M'$ between two objects such that $\psi(M)$ contains a Cauchy surface of M' . Consider now any $f \in \Omega_{0,\delta}^1(M')$ and choose $\chi \in C^\infty(M')$ such that $\chi = 1$ in $J^+(\psi(M); M') \setminus \psi(M)$ and $\chi = 0$ in $J^-(\psi(M); M') \setminus \psi(M)$. If we define $h \doteq f - \delta d(G^- f - \chi G f) = \delta d(\chi G f)$, we have identified a compactly supported 1-form h in $\psi(M)$ which differs from f by an element in $\mathcal{J}(M')$. Since $\Omega_{0,\delta}^1(M')$ generates $\mathcal{A}^0(M')$, this suffices to conclude the proof. \square

To conclude this section, we would like to remark once more that all proven statements rely heavily on the assumptions on the topology of the underlying background M . Most notably, if neither $H^1(M)$ nor $H^2(M)$ are trivial, definition 3.9 would not be fully consistent since the ideal $\mathcal{J}(M)$ is constructed out of the causal propagator respectively the symplectic form (cf. (3.4)). As we already pointed out, the latter would depend on the representative of the chosen equivalence class of gauge equivalent Lorenz solutions of (3.3). It is worth mentioning that this does not entail that it is utterly impossible to associate a suitable $*$ -algebra to the vector potential in spacetimes of non-trivial topology. A potential way out would be to mimic the approach of [11, 28] for the field strength tensor according to which the topological obstructions could be circumvented by associating to the underlying physical system the so-called universal algebra. We will not give further details here, but we want to stress that, although this approach is feasible, there is a heavy price to pay: As shown in [11, 28] for the case of the field strength, the theory cannot be described within the framework of general local covariance. On account of the generality of the argument given in these papers, we expect that the same result applies to the case of the vector potential. General local covariance is, however, not a

feature we are willing to give up in this paper.

4 Hadamard States

This section will be fully devoted to the analysis of the second step in the algebraic quantization scheme, namely the choice of a suitable (algebraic) state. Let us recall that this is a continuous linear functional $\omega : \mathcal{A}(M) \rightarrow \mathbb{C}$ such that $\omega(e) = 1$ and $\omega(a^*a) \geq 0$, where e is the algebra unit while a is an arbitrary element of $\mathcal{A}(M)$. Notice that here $\mathcal{A}(M)$ could stand for an arbitrary unital topological $*$ -algebra and not necessary for the one from definition 3.9. The GNS theorem entails that the choice of ω is tantamount to the identification of a triple $(\mathcal{D}_\omega, \pi_\omega, \Omega_\omega)$ where \mathcal{D}_ω is a dense subset of an Hilbert space \mathcal{H}_ω and $\pi_\omega : \mathcal{A}(M) \rightarrow \mathcal{L}(\mathcal{D}_\omega)$ is a representation of the algebra in terms of linear operators acting on \mathcal{D}_ω . Furthermore, Ω_ω is a norm 1 vector in \mathcal{D}_ω such that $\mathcal{D}_\omega = \{\pi_\omega(a)\Omega, \forall a \in \mathcal{A}(M)\}$. From now on we will further restrict our attention to the subclass of possible states which are quasi-free, *i.e.*, all n -point correlation functions ω_n can be constructed out of the two-point one, ω_2 , that is $\omega_n = 0$ for n odd, whereas if n is even

$$\omega_n(f_1 \otimes \cdots \otimes f_n) = \sum_{\pi_n \in S_n} \prod_{i=1}^{n/2} \omega_2(f_{\pi_n(2i-1)} \otimes f_{\pi_n(2i)}), \quad (4.1)$$

where $f_i \in \mathcal{A}(M)$, $i = 1, 2, \dots, n$ and where S_n denotes the ordered permutations of n elements, namely π_n fulfils that $\pi_n(2i-1) < \pi_n(2i)$ and $\pi_n(2i-1) < \pi_n(2i+1)$ for all $1 \leq i \leq n/2$.

The main goal of our investigation is to develop a procedure to identify in between the plethora of possible quasi-free states those which are physically meaningful. On curved backgrounds, it is nowadays widely accepted that such assertion coincides with asking that ω must be of Hadamard form, *i.e.*, it must satisfy a condition on the singular structure¹ of ω_2 . Although from a mathematical point of view *Hadamard states* can be characterized via the sophisticated tools of microlocal analysis as we shall discuss later, they are also noteworthy from a physical point of view. As a matter of fact their ultraviolet behaviour mimics that of the Minkowski vacuum and the quantum fluctuations of all observables, such as, for example, the smeared components of the stress-energy tensor are bounded, two conditions which are necessary for any state to be reasonably called physically sensible.

From a more practical perspective the study of Hadamard existence is commonly divided in two distinct problems: existence and explicit construction. If we focus on the specific case of the vector potential and its field algebra, as per definition 3.9, the first of these two problems has been already partly tackled. As first proven in [17] employing a deformation technique first introduced in [19] and under the additional hypothesis that the underlying background (M, g) is a globally hyperbolic spacetime with compact Cauchy surface Σ , Hadamard states do exist. Yet, even though it is easily conceivable that such a result could be extended by removing the topological assumption on Σ , the deformation procedure cannot yield a mean to concretely construct a Hadamard state on non-trivial backgrounds. This is instead the aim of this section and, as anticipated

¹The identification of the Hadamard form with a specific condition on the structure of the two-point function is actually valid regardless of the assumption that ω is quasi-free as first proven in [38].

in the introduction, we shall achieve our goal on those asymptotically flat spacetimes (M, g) which are genuine objects of \mathfrak{Man}' . The tool we shall use is the bulk-to-boundary reconstruction technique which we will now develop in detail for the vector potential and which has been already successfully applied to scalar and Dirac fields on asymptotically flat and cosmological spacetimes [9, 13, 14, 15] as well as on Schwarzschild spacetime to construct the Unruh state [12]. Notice that a more detailed account of the analysis presented in the next section is also available in [39].

4.1 The Projection to the Boundary

On account of the analysis of subsection 2.4, every asymptotically flat spacetime (M, g) comes endowed with a natural codimension 1 null differentiable boundary \mathcal{I}^+ which has the property of being intrinsic and universal. This suggests both that we should use null infinity as the screen on which to encode the information of a bulk field theory and that a boundary theory has to be defined in such a way that it does not depend on the chosen bulk (M, g) . Henceforth, let (M, g) be an asymptotically flat spacetime as per subsection 2.3 and let us call ψ the embedding into the unphysical spacetime $(\widetilde{M}, \widetilde{g})$. Moreover, let $\iota : \mathcal{I}^+ \hookrightarrow \widetilde{M}$ be the smooth embedding of null infinity into the unphysical spacetime. Starting from these data we get the following result:

Lemma 4.1. *Let (M', g') be another asymptotically flat spacetime diffeomorphic to an open subset of $(\widetilde{M}', \widetilde{g}')$ such that $\iota' : \mathcal{I}^+ \hookrightarrow \widetilde{M}'$ is a smooth embeddings of null infinity in \widetilde{M}' . Then the pull-back bundle $E_{\mathcal{I}} \doteq \iota^*(T^*\widetilde{M})$ is universal in the sense that there always exists a diffeomorphism $\psi : \mathcal{I}^+ \rightarrow \mathcal{I}^+$ such that $E'_{\mathcal{I}} = \psi^*E_{\mathcal{I}}$ with $E'_{\mathcal{I}} \doteq \iota'^*(T^*\widetilde{M}')$.*

Proof. Since all spacetimes we consider are globally hyperbolic, we know that they are parallelizable and thus their tangent and cotangent bundles are trivial [21, 22]. Therefore, $T^*\widetilde{M}$ is isomorphic to $\widetilde{M} \times \mathbb{R}^4$ and the pulled-back bundle on \mathcal{I}^+ is also trivial and isomorphic to $\mathcal{I}^+ \times \mathbb{R}^4$. The same result applies to \widetilde{M}' and thus the universality of null infinity suffices to prove the lemma because this latter property can be trivially lifted to $\mathcal{I}^+ \times \mathbb{R}^4$. \square

We can thus consider $E_{\mathcal{I}}$ and use it as the building block to associate to the boundary an intrinsic $*$ -algebra which encodes the bulk data. To this avail, we still lack an important ingredient: Pulling back the metric pairing $\widetilde{g}^{-1}(\cdot, \cdot)$ on the cotangent bundle of the unphysical spacetime to the boundary, we obtain a natural pairing on the bundle $E_{\mathcal{I}}$. To wit, we define on $E_{\mathcal{I}}$ the (degenerate) pairing $(\cdot, \cdot)_{\mathcal{I}} : E_{\mathcal{I}} \times E_{\mathcal{I}} \rightarrow \mathbb{R}$, such that

$$\xi_B^2(s, s')_{\mathcal{I}} \doteq \iota^*\widetilde{g}^{-1}(\omega, \omega'), \quad (4.2)$$

where $\omega, \omega' \in \Omega^1(\widetilde{M}) = \Gamma(T^*\widetilde{M})$ such that they induce via pull-back $s, s' \in \Gamma(E_{\mathcal{I}})$ respectively. We remind the reader that ξ_B is the very same function introduced in section 2.4 to rescale the conformal factor in such a way that the boundary structure admits a Bondi coordinate system $(u, \Xi, \theta, \varphi)$. Henceforth, we shall only work in this particular frame which is noteworthy because it induces on null infinity a natural measure $\mu_{\mathcal{I}} \doteq \sin^2\theta du d\theta d\varphi$. Accordingly, we can also introduce the pairing

$$\langle s, s' \rangle_{\mathcal{I}} \doteq \int_{\mathcal{I}^+} (s, s')_{\mathcal{I}} \mu_{\mathcal{I}}$$

for all sections $s, s' \in \Gamma(E_{\mathcal{J}})$ for which the integral converges. Using the sections of $E_{\mathcal{J}}$ and the above pairing, we can now construct a symplectic space:

Lemma 4.2. *The set*

$$\mathcal{S}(\mathcal{J}) \doteq \frac{\{f \in \Gamma(E_{\mathcal{J}}) \mid 0 \leq \langle f, f \rangle_{\mathcal{J}} < \infty \text{ and } 0 \leq \langle \partial_u f, \partial_u f \rangle_{\mathcal{J}} < \infty\}}{\{f \in \Gamma(E_{\mathcal{J}}) \mid \langle f, f \rangle_{\mathcal{J}} = 0 \text{ or } \langle \partial_u f, \partial_u f \rangle_{\mathcal{J}} = 0\}}$$

is a symplectic space invariant under the representation Π^{-2} of the BMS group (cf. (2.4)) if endowed with the following weakly non-degenerate symplectic form $\varsigma : \mathcal{S}(\mathcal{J}) \times \mathcal{S}(\mathcal{J}) \rightarrow \mathbb{R}$ such that

$$\varsigma(f, h) \doteq \langle f, \partial_u h \rangle_{\mathcal{J}} - \langle \partial_u f, h \rangle_{\mathcal{J}} \quad (4.3)$$

for any $f, h \in \mathcal{S}(\mathcal{J})$.

Proof. As a first step we notice that the right hand side of (4.3) is per definition of $\mathcal{S}(\mathcal{J})$ well defined. Moreover, by direct inspection the symplectic form is both bilinear and antisymmetric. Employing the Cauchy-Schwarz inequality, we see that ς is indeed weakly non-degenerate per definition of $\mathcal{S}(\mathcal{J})$ and the trivial kernel of ∂_u acting on elements of $\mathcal{S}(\mathcal{J})$.

Concerning BMS invariance, let us consider a generic element $\gamma \in BMS$. According to the translation invariance of the measure, it holds

$$\varsigma(f, h) \mapsto \int_{\mathcal{J}^+} K_{\gamma}^2 (K_{\gamma}^{-2} f, K_{\gamma}^{-1} \partial_u (K_{\gamma}^{-2} h))_{\mathcal{J}} K_{\gamma}^3 \mu_{\mathcal{J}} = \varsigma(f, h),$$

which is indeed the sought property. \square

Since $(f, f)_{\mathcal{J}}$ with $f \in \mathcal{S}(\mathcal{J})$ is integrable, it turns out that $\lim_{u \rightarrow \pm\infty} f = 0$ as one can establish with a minor readaptation of the proof in [30, Footnote 7]. Therefore, we can integrate ς by parts and rewrite it as

$$\varsigma(f, h) = 2 \langle f, \partial_u h \rangle_{\mathcal{J}}.$$

We can employ this symplectic space to associate an intrinsic $*$ -algebra to null infinity:

Definition 4.3. We call **boundary algebra** $\mathcal{A}(\mathcal{J})$ the $*$ -algebra realized as the quotient between the free unital $*$ -algebra $\mathcal{A}(\mathcal{J})^0 \doteq \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{S}(\mathcal{J})^{\otimes n} \otimes \mathbb{C}$ and the $*$ -ideal generated by the elements of the form $-i\varsigma(f, h) \oplus f \otimes h - h \otimes f$ for all $f, h \in \mathcal{S}(\mathcal{J})$. As usual, the $*$ -operation is defined as complex conjugation.

Having established an abstract boundary theory, we will now bring to attention a striking relationship between the Maxwell theory in the bulk and the theory on null infinity. To this avail, let us first individuate a certain gauge in the space residual gauge transformations of Lorenz solutions. In the following propositions it will often be convenient to consider (in the unphysical spacetime) an arbitrary coordinate system centred at i^+ with a coordinate U which is the affine parameter along the past-directed null geodesics originating from future timelike infinity.

Proposition 4.4. For each $[f] \in \mathcal{S}(M)$ there exists a representative $f \in \Omega_{0,\delta}^1(M)$ such that the u -component of the pulled-back solution in the unphysical spacetime is vanishing at \mathcal{J}^+ : $(\iota^* \tilde{G}^- \psi_*^{-2} f)(\partial_u) = 0$.² We say that f is in the \mathcal{J} -gauge.

² $(\iota^* \tilde{G}^- \psi_*^{-2} f)(\partial_u)$ denotes the contraction of the pulled-back solution with the vector field ∂_u .

Proof. Let $f \in \Omega_{0,\delta}^1(M)$ be an arbitrary representative of $[f]$ and let $\tilde{f} = \psi_*^{-2}f$. According to the characteristic initial value problem there exists to each $\chi \in C^\infty(\mathcal{I})$ a $\lambda \in C^\infty(\widetilde{M})$ such that $\tilde{\square}\lambda = 0$ in $J^-(i^-; \widetilde{M})$ and $\lambda|_{\mathcal{I}^+} = \chi$. Thus, setting $\chi = \int_{-\infty}^U (\tilde{G}^- \tilde{f})(n) dU'$, where n is the tangent vector field generating the null geodesics along \mathcal{I}^+ , there exists $\tilde{f}' \in \Omega_{0,\delta}^1(\psi(M))$ such that $\tilde{G}\tilde{f}' = \tilde{G}\tilde{f} - d\lambda$ by the uniqueness of the causal propagator. Then, by corollary 3.3, there exists also $\tilde{\alpha} \in \Omega_0^1(\psi(M))$ such that $\tilde{f}' = \tilde{f} - \tilde{\delta}d\tilde{\alpha}$ and thus, applying (2.1c), we conclude that $f' = \Xi^{-2}\psi^*\tilde{f}' \in [f]$.

Furthermore, employing the invariance of the volume under pull-backs, we see that

$$\iota^* d \int_{-\infty}^U (\tilde{G}^- \tilde{f})(n) dU' = d \int_{\mathbb{R}} (\iota^* \tilde{G}^- \tilde{f})(\partial_u) du'$$

because n and dU have opposite conformal weight. Hence, we obtain for the u -component

$$(\iota^* \tilde{G}^- \tilde{f}')(\partial_u) = (\iota^* \tilde{G}^- \tilde{f})(\partial_u) - \partial_u \int_{-\infty}^u (\iota^* \tilde{G}^- \tilde{f})(\partial_u) du' = 0$$

by the fundamental theorem of calculus. \square

Note that this residual gauge transformation fixes the gauge completely because any further gauge transformation would necessarily be constant along the u -direction and compactly supported at \mathcal{I}^+ and thus identically zero by the uniqueness of the characteristic initial value problem.

Let us now define a map $b : \mathcal{S}(M) \rightarrow \mathcal{S}(\mathcal{I})$ which we will show to be a symplectomorphism from the bulk theory $(\mathcal{S}(M), \sigma)$ to the boundary theory $(\mathcal{S}(\mathcal{I}), \varsigma)$ in the next theorem.

Theorem 4.5. *Let $b : \mathcal{S}(M) \rightarrow \mathcal{S}(\mathcal{I})$ be the map defined by*

$$b([f]) = \iota^* \tilde{G}^- \psi_*^{-2} f,$$

where f is a representative of $[f]$ in the \mathcal{I} -gauge. Then b is a symplectomorphism, i.e., for all $[f], [h] \in \mathcal{S}(M)$ it holds that

$$\sigma([f], [h]) = \varsigma(b([f]), b([h])),$$

*where σ is given in (3.4) and ς in (4.3). b is dubbed the **bulk-to-boundary projection**.*

Proof. First, we show that $b([f]) \in \mathcal{S}(\mathcal{I})$, i.e., $0 \leq \langle b([f]), b([f]) \rangle_{\mathcal{I}} < \infty$ and $0 \leq \langle \partial_u b([f]), \partial_u b([f]) \rangle_{\mathcal{I}} < \infty$. Since the advanced propagator \tilde{G}^- is supported in the forward light cone, we have that $(\text{supp } \tilde{G}^- \psi_*^{-2} f) \cap J^-(i^+; \widetilde{M})$ is a compact set for all $f \in \Omega_0^1(M)$ by the global hyperbolicity of \widetilde{M} . Hence, the restriction to \mathcal{I}^+ of $\tilde{G}^-(\psi_*^{-2} f)$ lies in $C_0^\infty(\mathcal{I}^+ \cup i^+)$ and there exists u' such that $b([f])(u, \theta, \varphi) = 0$ for all $u > u'$. It follows that $\langle b([f]), b([f]) \rangle_{\mathcal{I}} < \infty$ by invariance of the volume under pull-backs and cancelation of the conformal weight of the volume form $\mu_{\mathcal{I}}$ and the inner product $(\cdot, \cdot)_{\mathcal{I}}$. Moreover, the vanishing u -component of $b([f])$ guarantees that $(b([f]), b([f]))_{\mathcal{I}}$ and $(\partial_u b([f]), \partial_u b([f]))_{\mathcal{I}}$ are everywhere positive by the definition of the inner product in (4.2) and therefore that $\langle b([f]), b([f]) \rangle_{\mathcal{I}} \geq 0$ and $\langle \partial_u b([f]), \partial_u b([f]) \rangle_{\mathcal{I}} \geq 0$. If we consider again a coordinate system centred at i^+ all first derivatives of $(\tilde{G}^- \psi_*^{-2} f)$ are still bounded and compactly supported functions on $\mathcal{I}^+ \cup i^+$. Yet, since we want to work in a Bondi frame and we are interested in $\partial_u b(f)$, we need to keep in mind that, on account of the geometric

defining properties of asymptotically flat spacetimes, $\partial_u b([f]) = \xi_B^{-1} \iota^* \partial_U (\tilde{G}^- \psi_*^{-2} f) \upharpoonright_{\mathcal{I}^+}$. We can now exploit the estimate on the fall-off behaviour of the Bondi factor towards i^+ as in [29, Lemma 4.4] together with the boundedness of $\partial_U (\tilde{G}^- \psi_*^{-2} f) \upharpoonright_{\mathcal{I}^+}$ to conclude that $\langle \partial_u b([f]), \partial_u b([f]) \rangle_{\mathcal{I}} < \infty$. This entails that $b([f]) \in \mathcal{S}(\mathcal{I})$.

In order to prove that b is a symplectomorphism, let us recall that the symplectic form of the vector potential is conformally invariant (*c.f.* corollary 3.7) to rewrite (3.4) as

$$\sigma([f], [h]) = \int_M Gf \wedge *h = \int_{\tilde{M}} \tilde{G}(\psi_*^{-2} f) \wedge \tilde{*}(\psi_*^{-2} h) = \int_{\tilde{M}} \tilde{G}(\psi_*^{-2} f) \wedge \tilde{*} \tilde{\delta} d\tilde{G}^-(\psi_*^{-2} h),$$

where, in the last equality, we employed implicitly the coclosedness of $\psi_*^{-2} h$ and the defining properties of the retarded fundamental solution. Choosing the representatives f, h such that $(\iota^* \tilde{G}^- \psi_*^{-2} f)(\partial_u) = 0$ and $(\iota^* \tilde{G}^- \psi_*^{-2} h)(\partial_u) = 0$ and applying Green's identity, we conclude that

$$\begin{aligned} \sigma([f], [h]) &= \int_{\mathcal{I}^+} \iota^* (\tilde{G}^-(\psi_*^{-2} h) \wedge \tilde{*} d\tilde{G}^-(\psi_*^{-2} f) - \tilde{G}^-(\psi_*^{-2} f) \wedge \tilde{*} d\tilde{G}^-(\psi_*^{-2} h)) \\ &= \langle b([f]), \partial_u b([h]) \rangle_{\mathcal{I}} - \langle \partial_u b([f]), b([h]) \rangle_{\mathcal{I}}. \end{aligned} \quad \square$$

We have thus proven that b is a symplectomorphism between $(\mathcal{S}(M), \sigma)$ and $(\mathcal{S}(\mathcal{I}), \varsigma)$ which, exactly as it happened in [13], induces a natural injective $*$ -homomorphism between the bulk and the boundary algebra. Denoting this $*$ -homomorphism by the same symbol, we thus obtain:

Lemma 4.6. *The symplectomorphism b induces an injective $*$ -algebra homomorphism $b : \mathcal{A}(M) \rightarrow \mathcal{A}(\mathcal{I}^+)$ such that*

$$b(A_M(f)) \doteq A_M(b([f])),$$

for all $[f] \in \mathcal{S}(M)$. Accordingly, every algebraic state $\omega^{\mathcal{I}} : \mathcal{A}(\mathcal{I}^+) \rightarrow \mathbb{C}$ on the boundary algebra induces a state $\omega^M : \mathcal{A}(M) \rightarrow \mathbb{C}$ on the bulk field algebra unambiguously defined by $\omega^M \doteq \omega^{\mathcal{I}} \circ b$.

4.2 Hadamard States Induced from Null Infinity

As we have already remarked at the beginning of this section, our goal is to provide a scheme to construct Hadamard states on a non-trivial background (M, g) . The traditional techniques rely on g possessing a complete timelike Killing field, that is M is stationary, so that it possible to make sense of the notion of positive energy out of a Fourier transform. Clearly, this procedure is doomed to failure in the most general scenario since the group of isometries of (M, g) can even be trivial. The last lemma of the previous subsection provides us with potential way to circumvent such an obstruction by inducing a state for the bulk field theory out of one for the boundary counterpart. This point of view has an a priori advantage due to the geometry of \mathcal{I}^+ which possesses a natural direction, identified by the u -coordinate in a Bondi frame, along which one can perform a Fourier transform. This direction plays for the boundary the same role as the time direction in a stationary or static spacetime, thus allowing us to single out a natural and distinguished algebraic state for the boundary theory [13, 29, 30]. We will now show that this line of reasoning can also be applied to the case of the vector potential. As a starting point we focus on the boundary $*$ -algebra:

Proposition 4.7. *The map $\omega_2^{\mathcal{J}} : \mathcal{S}(\mathcal{J}) \otimes \mathcal{S}(\mathcal{J}) \rightarrow \mathbb{R}$ such that*

$$\omega_2^{\mathcal{J}}(f \otimes h) \doteq \lim_{\epsilon \rightarrow 0} -\frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{((f(u, \theta, \varphi), h(u', \theta, \varphi))_{\mathcal{J}})}{(u - u' - i\epsilon)^2} du du' d\mathbb{S}^2(\theta, \varphi), \quad (4.4)$$

where $d\mathbb{S}^2(\theta, \varphi) = \sin^2 \theta d\theta d\varphi$ is the standard measure on the unit 2-sphere, unambiguously identifies a quasi-free state $\omega^{\mathcal{J}} : \mathcal{A}(\mathcal{J}) \rightarrow \mathbb{C}$. Furthermore,

- a) $\omega^{\mathcal{J}}$ induces a bulk state $\omega^M \doteq \omega^{\mathcal{J}} \circ b : \mathcal{A}(M) \rightarrow \mathbb{C}$ which is quasi-free and
- b) $\omega^{\mathcal{J}}$ is invariant under the action of the BMS group $\rho : \mathcal{A}(\mathcal{J}) \rightarrow \mathcal{A}(\mathcal{J})$ induced by (2.3), viz. $\omega^{\mathcal{J}} \circ \rho = \omega^{\mathcal{J}}$.

Proof. Since the elements f of $\mathcal{S}(\mathcal{J})$ are such that $\langle f, f \rangle_{\mathcal{J}} < \infty$ and $\langle \partial_u f, \partial_u f \rangle_{\mathcal{J}} < \infty$, barring a minor readjustment, we can slavishly employ the analysis of [14, 15] to conclude that β_2 identifies a well-defined two-point distribution on null infinity. Furthermore, since we know that both f and h tend to 0 as u diverges, we can apply integration by parts, Parseval's identity and the convolution theorem³ to obtain

$$\omega_2^{\mathcal{J}}(f \otimes h) = \frac{1}{\pi} \int_{\mathbb{R} \times \mathbb{S}^2} k \Theta(k) (\widehat{f}(k, \theta, \varphi), \widehat{h}(-k, \theta, \varphi))_{\mathcal{J}} dk d\mathbb{S}^2(\theta, \varphi), \quad (4.5)$$

where $\Theta(k)$ is the Heaviside step function. Hence, $\omega_2^{\mathcal{J}}(f \otimes \bar{f}) \geq 0$ and the corresponding state $\omega^{\mathcal{J}}$ satisfies positivity. A direct calculation moreover shows that $\omega_2^{\mathcal{J}}(f \otimes h) - \omega_2^{\mathcal{J}}(h \otimes f) = i\zeta(f, h)$ and thus $\omega_2^{\mathcal{J}}$ unambiguously identifies an algebraic quasi-free state for $\mathcal{A}(\mathcal{J})$.

Concerning BMS invariance, it is sufficient to prove that it holds true for the two-point function as the state is quasi-free. Therefore, the natural representation of a BMS transformation $\gamma \in BMS$ on $\mathcal{S}(\mathcal{J})$ via Π_{γ}^{-2} yields

$$(\omega_2^{\mathcal{J}} \circ \rho)(f \otimes h) = \lim_{\epsilon \rightarrow 0} -\frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{K_{\gamma}^2(K_{\gamma}^{-2}f(u, \theta, \varphi), K_{\gamma}^{-2}h(u', \theta, \varphi))_{\mathcal{J}}}{(K_{\gamma}u - K_{\gamma}u' - i\epsilon)^2} K_{\gamma}^4 du du' d\mathbb{S}^2(\theta, \varphi),$$

where we suppressed the explicit dependence of the conformal factor $K_{\gamma}(\theta, \varphi)$ on the angular variables. Furthermore, since this factor is bounded, the right hand side coincides with $\omega_2^{\mathcal{J}}(f \otimes h)$ up to an irrelevant redefinition of the ϵ -parameter. \square

Although we have identified a BMS-invariant state on \mathcal{J}^+ , we are still far from claiming that its bulk counterpart is physically meaningful. In order to answer this question, first of all we need to make precise the concept of Hadamard states. Suppose for now that we are equipped with an arbitrary but quasi-free algebraic state $\lambda : \mathcal{A}(M) \rightarrow \mathbb{C}$ with associated two-point function $\lambda_2(f \otimes h)$ for all $f, h \in \Omega_{0, \delta}^1(M)$. From a mathematical point of view the Hadamard condition is a statement concerning the singular structure of the bi-distribution λ_2 and it is thus best studied by means of the techniques proper of microlocal analysis. These are presented in the monograph [27] whose notations, definitions and conventions we will adopt here. Since in our scenario we cope with vector bundles instead of scalar functions, we need to slightly extend the ordinary notion of the

³The conventions for the Fourier transform are as in [27].

wavefront set. As noted after [27, Th. 8.2.4], the wavefront set $\text{WF}(u)$ of a distribution $u \in \mathcal{D}'(E)$ on a vector bundle E is defined locally as $\cup_i \text{WF}(u_i)$, where (u_1, \dots, u_N) are the components of u in a local trivialization of E . This definition is indeed invariant under a change of the local trivialization and thus yields a sensible extension of the wavefront set to distributional sections. Hence, we can follow [34, 35, 37]:

Definition 4.8. *Let λ be an algebraic state on $\mathcal{A}(M)$. We say that λ fulfils the **Hadamard condition**, and is thus called a **Hadamard state**, if its two-point function λ_2 satisfies*

$$\text{WF}(\lambda_2) = \{(x, y, k_x, -k_y) \in T^*M^2 \setminus 0 \mid (x, k_x) \sim (y, k_y), k_x \triangleright 0\},$$

where 0 is the zero section of T^*M^2 . Here $(x, k_x) \sim (y, k_y)$ means that the point x and y are connected by a null geodesic γ so that k_x is cotangent to it in x and k_y is the parallel transport along γ of k_x from x to y . Furthermore, $k_x \triangleright 0$ means that the covector k_x is future-pointing.

Therefore, we have to confirm that the wavefront set of the two-point function of the bulk state induced from (4.4) has indeed the form given in definition 4.8. Let us notice that ω_2^M satisfies an additional property of the two-point function of a state of $\mathcal{A}(M)$:

$$\omega_2^M(\square f \otimes h) = \omega_2^M(f \otimes \square h) = 0$$

for all $f, h \in \Omega_{0,\delta}^1(M)$, i.e. ω_2^M is a bi-solution of the equations of motion (3.3). Thanks to this, we can prove the main theorem of this section:

Theorem 4.9. *The state $\omega^M \doteq \omega^\mathcal{J} \circ b : \mathcal{A}(M) \rightarrow \mathbb{C}$ where $\omega^\mathcal{J} : \mathcal{A}(\mathcal{J}) \rightarrow \mathbb{C}$ is the unique quasi-free state built out of (4.4)*

- a) *is of Hadamard form,*
- b) *is invariant under the action of all isometries of (M, g) , that is $\omega^M \circ \alpha_\phi = \omega^M$ for all possible isometries ϕ of (M, g) . Here α is the $*$ -isomorphism representation of the isometry group on $\mathcal{A}(M)$ defined by the action on the algebra generators $f \in \Omega_{0,\delta}^1$, i.e., $\alpha_\phi(f) \doteq \phi_* f$.*
- c) *coincides with the Poincaré vacuum if the bulk spacetimes is Minkowski.*

Proof. Let us start from a). Since β_M is a quasi-free state, we focus only on ω_2^M . In order to prove the Hadamard property, it is convenient to work with the auxiliary object

$$\widetilde{\omega}_2^M(f \otimes h) \doteq \omega_2^\mathcal{J}(\iota^* \tilde{G}^- f \otimes \iota^* \tilde{G}^- h),$$

where $f, h \in \Omega_0^1(\widetilde{M})$. Notice that this is a well-defined expression since $J^+(\text{supp}(f + h); \widetilde{M}) \cap J_{\widetilde{M}}^-(i^+)$ is compact and that, on account of the support properties of the retarded fundamental solution of the Laplace-de Rham operator, $\text{supp } \omega_2^{\widetilde{M}} \subseteq J^-(i^+; \widetilde{M}) \setminus \{i^-\}$. Furthermore, it holds

$$\omega_2^{\widetilde{M}}(\square f \otimes \square h) = \omega_2^\mathcal{J}(\iota^* f \otimes \iota^* h).$$

Hence, the wavefront set of $(\square \otimes \square) \omega_2^{\widetilde{M}}$ coincides with that of $\omega_2^\mathcal{J}$ which can be calculated directly from (4.4) and, following almost slavishly [15, 30],

$$\text{WF}(\omega_2^\mathcal{J}) = \{(x, x, k, -k) \in T^*(\mathcal{J}^+)^2 \setminus 0 \mid k(\partial_u) > 0\},$$

where $k(\partial_u)$ can be read as the component of the covector k along the u -direction. If we apply the theorem of propagation of singularities, we can obtain

$$\begin{aligned} \text{WF}(\omega_2^{\widetilde{M}}) = \{ & (x, y, k_x, -k_y) \in T^*\widetilde{M}^2 \setminus 0 \mid \exists z \in \mathcal{I}^+, k_z \in T_z^*\widetilde{M} \text{ such that} \\ & x, y \in J_M^-(i^+) \setminus \{i^-\}, (x, k_x) \sim (y, k_y) \sim (z, k_z), k_z(\partial_u) > 0\}. \end{aligned}$$

To conclude, we need to restrict our test functions to those which are compactly supported in $\psi(M)$, the image of M in \widetilde{M} . Then, on account of the invariance of null geodesics under conformal transformations and of the fact that there does not exist a null geodesic joining $x \in \psi(M)$ with i^+ [30, Lemma 4.3], we can conclude that the wavefront set of ω_2^M has the same form as that of definition 4.8. Thus ω_M is a Hadamard state.

Let us now prove b). The same argument given in [30, Th. 3.1] guarantees that the statement holds true if proven for all one-parameter groups of isometries ϕ_t^X , $t \in \mathbb{R}$ induced by a Killing vector field X . On account of proposition 4.7, $\omega^M \circ \alpha_{\phi_t^X} = \omega^{\mathcal{I}} \circ b \circ \alpha_{\phi_t^X}$. Furthermore, it holds [30, Prop. 3.4] that $b \circ \alpha_{\phi_t^X} = \rho_{\tilde{\phi}_t^{\tilde{X}}} \circ b$, where $\rho_{\tilde{\phi}_t^{\tilde{X}}}$ is the action on $\mathcal{A}(\mathcal{I})$ of a one-parameter subgroup of the BMS induced via the exponential map from \tilde{X} , the unique extension of \tilde{X} to the boundary. Since in proposition 4.4 we proved the invariance of $\omega^{\mathcal{I}}$ under the action of the BMS group,

$$\omega^M \circ \alpha_{\phi_t^X} = \omega^{\mathcal{I}} \circ \rho_{\tilde{\phi}_t^{\tilde{X}}} \circ b = \omega^{\mathcal{I}} \circ b = \omega^M.$$

Having already established both a) and b), we can conclude that, in Minkowski spacetime, ω^M is a quasi-free, Hadamard state which is Poincaré invariant. Either arguing via the uniqueness of the ground state in flat spacetime or repeating the proof of [13, Th. 4.1], we establish c). \square

An additional advantage of inducing states for the bulk field algebra from the boundary counterpart is the possibility to define a KMS state for the boundary theory on \mathcal{I}^+ . This was first introduced in [12] to rigorously define the Unruh state on Schwarzschild spacetime and it was later applied to scalar and Dirac fields on a certain class of cosmological spacetime in [9]. The rationale behind the whole procedure is that the boundary state can be constructed starting from a two-point function which admits an explicit mode decomposition and which can be easily modified by adding a Bose factor to construct a KMS state. We will not dwell on the definition of such class of states and we refer to appendix B of [12] for an overview. Hence, in the case at hand, the starting point would be (4.5) out of which we define

$$\omega_{2,T}^{\mathcal{I}}(f \otimes h) = \frac{1}{\pi} \int_{\mathbb{R} \times \mathbb{S}^2} \frac{k}{1 - e^{-\frac{k}{T}}} (\hat{f}(k, \theta, \varphi), \hat{h}(-k, \theta, \varphi))_{\mathcal{I}} dk d\mathbb{S}^2(\theta, \varphi), \quad (4.6)$$

where $T \geq 0$ and $f, h \in \mathcal{S}(\mathcal{I})$.

Proposition 4.10. *The two-point function (4.6) induces a state $\omega_T^{\mathcal{I}} : \mathcal{A}(\mathcal{I}) \rightarrow \mathbb{C}$ which is quasi-free, enjoys the KMS property with respect to the u -translations on \mathcal{I}^+ and which converges weakly to the state $\omega^{\mathcal{I}}$ of proposition 4.7 as $T \rightarrow 0$.*

Proof. Since $0 \leq \omega_2^{\mathcal{I}}(f \otimes \bar{f}) < \omega_{2,T}^{\mathcal{I}}(f \otimes \bar{f})$, we can infer that a quasi-free state built out (4.1) is indeed positive. It is also rather straightforward to see that such a state satisfies the canonical commutation relations since, as we will explicitly write out in the

next lemma, the difference between (4.6) and the counterpart at $T = 0$ is symmetric. Furthermore, $\omega_T^\mathcal{J}$ is invariant under the translations generated by ∂_u , i.e. $f(u, \theta, \varphi) \mapsto \alpha_\lambda f(u, \theta, \varphi) \doteq f(u - \lambda, \theta, \varphi)$ and therefore $\widehat{f}(k, \theta, \varphi) \mapsto \widehat{f}(k, \theta, \varphi)e^{ik\lambda}$ for some $\lambda \in \mathbb{R}$ and any $f \in \mathcal{S}(\mathcal{J})$, as can be inferred by direct inspection of the explicit form of the two-point function (4.6). We are thus left with proving the KMS condition. Given $f, h \in \mathcal{S}(\mathcal{J})$ and taking the analytic continuation of f, h in the u -variable to the complex plane, we have

$$\omega_{2,T}^\mathcal{J}(f \otimes \alpha_{iT-1}h) = \omega_{2,T}^\mathcal{J}(h \otimes f)$$

which guarantees that the KMS property holds true for $\omega_T^\mathcal{J}$ [6, Sect. 5.3]. The weak convergence of β_T to β as $T \rightarrow 0$ is an immediate consequence of the explicit form of (4.6) and of the regularity properties of each $f, h \in \mathcal{S}(\mathcal{J})$ \square

If we follow lemma 4.6, we know that $\omega_T^\mathcal{J}$ induces a bulk state ω_T^M but, a priori, we cannot hope that it will be thermal unless (M, g) is a static spacetime, hence endowed with a timelike Killing field with respect to which we can define a KMS condition. Yet, as already outlined in [9], the bulk state has a reminiscence of the exact thermal structure on the boundary and thus one might use it as a natural candidate to discuss physical phenomena for which we would like to give an at least approximate or asymptotic thermal interpretation. To this avail, we need, nonetheless, to make sure that ω_T^M is physically meaningful and as a last task of this section we prove

Proposition 4.11. *The state $\omega_T^M \doteq \omega_T^\mathcal{J} \circ b : \mathcal{A}(M) \rightarrow \mathbb{C}$ is a quasi-free, Hadamard state.*

Proof. Since $\omega_T^\mathcal{J}$ is a quasi-free state, ω_T^M enjoys, per construction, the very same property. In order to prove that it is Hadamard, we can focus on the associated two-point function $\omega_{2,T}^M$ and we introduce as an auxiliary tool

$$\Delta_T(f \otimes h) \doteq \omega_{2,T}^M(f \otimes h) - \omega_2^M(f \otimes h),$$

for all $f, h \in \Omega_{0,\delta}^1(M)$. In term of modes this last expression reads:

$$\Delta_T(f, h) = \frac{1}{\pi} \int_{\mathbb{R} \times \mathbb{S}^2} \frac{|k|}{e^{\frac{|k|}{T}} - 1} (\widehat{b(f)}(k, \theta, \varphi), \widehat{b(h)}(-k, \theta, \varphi))_\mathcal{J} dk d\mathbb{S}^2(\theta, \varphi),$$

where b is the bulk-to-boundary projection introduced in theorem 4.5. Since the prefactor $|k|/(e^{|k|/T} - 1)$ is bounded and both $b(f)$ and $b(h)$ are square-integrable on \mathcal{J}^+ as per theorem 4.5, $\Delta_T(f, h)$ is a well-defined bi-distribution. Moreover, the prefactor decays faster than any power of $|k|$ and, thus, if (x, y, k_x, k_y) is a point in $\text{WF}(\Delta_T)$, k_x and k_y must have vanishing component along causal directions. As Δ_T is also a bi-solution of the dynamics (3.3), the propagation of singularities theorem already implies that Δ_T has vanishing wavefront set and thus must be smooth. Therefore, ω_T^M is a Hadamard state. \square

5 Conclusions

The main goal of this paper is the explicit construction of physically sensible quantum states for the vector potential on a large class of curved backgrounds. Up to now, this has

been a rather elusive task on account of the gauge invariance of the underlying physical system which is known to create problems already in Minkowski spacetime.

As a first step we developed an algebraic quantization scheme for the vector potential on globally hyperbolic spacetimes. The underlying classical dynamics is ruled by a normally hyperbolic operator once we work with gauge equivalence classes. Thus, in principle, we could follow the standard procedure for bosonic field theories which calls for constructing a unital $*$ -algebra, the field algebra, associated to the space of solutions of the equations of motion. Yet, to this avail, one has to prove that such a space can be endowed with a weakly non-degenerate symplectic form. Extending the result of [8], we proved the existence of a well-defined symplectic form for the vector potential if either the first or the second de Rham cohomology group of the underlying background is trivial. Nevertheless, if these topological obstructions are not present, we have also shown that the vector potential can be interpreted as a locally covariant conformal quantum field theory in the sense of [32].

The second step of our construction has been the explicit construction of an algebraic state of Hadamard form. To this avail, we employed the bulk-to-boundary reconstruction technique on asymptotically flat spacetimes first introduced in [13]. This method calls for identifying the bulk algebra of observables as a $*$ -subalgebra of a second algebra, intrinsically built on null infinity. The advantage of this point of view is that each state on \mathcal{S}^+ automatically identifies a bulk counterpart. Furthermore, since \mathcal{S}^+ is a three-dimensional null differentiable manifold endowed with an infinite dimensional symmetry group, the BMS group, it is possible to identify a distinguished state $\omega^{\mathcal{S}}$ on the boundary algebra. The bulk counterpart ω^M turns out to enjoy several interesting properties. Most notably we proved that it is invariant under the action of all isometries and it is of Hadamard form. Furthermore, along the lines of [9], we have shown that it is possible to slightly modify the form of $\omega^{\mathcal{S}}$ to construct a family of states $\omega_T^{\mathcal{S}}$ which fulfil an exact KMS condition on \mathcal{S}^+ with respect to the translations along the null direction. Analogously to $\omega^{\mathcal{S}}$, also $\omega_T^{\mathcal{S}}$ induces for all $T \geq 0$ a bulk state ω_T^M . Although we cannot expect that ω_T^M is a KMS state unless the underlying background is static and thus possesses a timelike Killing field, it turns out that such a bulk state still enjoys the Hadamard property. Therefore, these states are natural candidates to deal on curved backgrounds with physical phenomena which admit a “thermodynamical” interpretation.

Recall that we used a complete gauge fixing procedure to construct a positive state. We expect no major obstacles repeating a similar bulk-to-boundary construction utilizing an indefinite metric ansatz similar to the Gupta-Bleuler formalism. A state constructed in such a manner might be useful in treating interacting quantum field theories perturbatively.

In terms of future perspectives, we envisage that our results could be used at least in two different directions. First of all, one can extend the whole construction to the same class of cosmological spacetimes studied in [9] and then discuss from a mathematically rigorous point of view the role of the vector potential in the analysis of semiclassical Einstein’s equation. This potential project is closely related to the second one, namely the construction of the extended algebra of Wick polynomials for the vector potential and the associated computation of the trace anomaly for the associated stress energy tensor along the lines of the same analysis for Dirac fields as in [10, 26].

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